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A linear equation for Wilson loops

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Abstract

The Makeenko–Migdal loop equation is non-linear and first order in the area derivative, but we show that for simple loops in QCD_2 it is possible to reformulate this equation as a linear equation with second order derivatives. This equation is a bound state Schrödinger equation with a three-dimensional Coulomb potential. Thus, loop dynamics leads to a surprising new picture of confinement, where this phenomenon is due to a (bound state) localization in loop space, with the Wilson loops decaying exponentially outside a characteristic radius.

The Makeenko–Migdal loop equation [1] is an exact consequence of the non-linear dynamics in $N = \infty$ QCD, and is therefore also non-linear. It is of first order in the area derivative. However, although the equation is closed, loop space is enormous, so it is not easy to gain insight into how dynamics look like in this space. In the present Letter we restrict ourselves to simple loops with an arbitrary number of windings n corresponding to n quarks propagating along the boundary of the loops. We then reformulate the loop equation for these simple loops in QCD_2 starting from the Makeenko–Migdal equation. The result is an equation which is linear but of second order in the derivatives. This is obtained from the Hopf equation in QCD_2, which can be generalized to a functional equation in higher dimensions, so the two-dimensional result may be generalizable to higher dimensions. In any case, our linear equation represents a new picture of confinement as a bound state in a three-dimensional space. The interaction between the n quarks is given by the three-dimensional Coulomb potential in loop space! This shows that the dynamics in loop space is rather surprising and is not in a simple correspondence with perturbative expansions.

For simple loops with a number n of windings it is possible to derive an equation for the spectral density of eigenvalues

$$\rho_C(\theta) = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{\infty} W^{(n)}(C)\cos(n\theta)\right], \tag{1}$$

where $W^{(n)}(C)$ is the Wilson loop for the simple curve C winding n times. The equation is [2]

$$\partial_\mu \frac{\delta\rho_C(\theta)}{\delta\sigma_{\mu\nu}(x)} = 2g_0^2 N \oint_C dy_\nu\, \delta^{(D)}(x-y) \times \frac{\partial}{\partial\theta}\left[\rho_C(\theta)\mathrm{P}\int_{-\infty}^{\infty} d\alpha \frac{\rho_C(\alpha)}{\theta-\alpha}\right], \tag{2}$$

where P means the principal value of the integral. If we introduce the resolvent (i.e., the Hilbert transform)

$$f_C(\theta) = \int_{-\infty}^{\infty} d\alpha \frac{\rho_C(\alpha)}{\theta-\alpha} = \mathrm{P}\int_{-\infty}^{\infty} d\alpha \frac{\rho_C(\alpha)}{\theta-\alpha} + i\pi\rho_C(\theta), \tag{3}$$

we can rewrite Eq. (2) as

$$\partial_\mu \frac{\delta f_C(\theta)}{\delta\sigma_{\mu\nu}(x)} = 2g_0^2 N \oint_C dy_\nu\, \delta^{(D)}(x-y) f_C(\theta)\frac{\partial f_C(\theta)}{\partial\theta}. \tag{4}$$

This equation is non-linear, as expected.

In two dimensions Eq. (4) reduces to the partial differential equation [2],

$$\frac{\partial f(A,\theta)}{\partial A} + f(A,\theta)\frac{\partial f(A,\theta)}{\partial\theta} = 0, \tag{5}$$

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where it was used that in $D = 2$ the Wilson loops depend only on the area A, which is measured in units of $g_0^2 N$. Eq. (5) is called the Hopf (or Burgers[1]) equation. In [3] it is shown that this type of equation, or a slight generalization, occurs for any multiplicatively free family of random variables [4]. Eq. (4) is valid in all dimensions and is a Hopf equation in loop space.

We shall now show that for the boundary conditions $W^{(n)}(0) = 1$ the non-linear first order Hopf equation (5) reduces to a linear second order equation. Let us use the notation $f(0,\theta) = f_0(\theta)$. By means of the method of characteristics Eq. (5) can be reformulated to

$$f(A,\theta) = f_0\big(\xi(A,\theta)\big), \quad \xi(A,\theta) = \theta - A f_0\big(\xi(A,\theta)\big), \tag{6}$$

and the various derivatives can be expressed as

$$\frac{\partial f(A,\theta)}{\partial A} = \frac{-f_0(\xi) f_0'(\xi)}{1 + f_0'(\xi) A}, \qquad \frac{\partial f(A,\theta)}{\partial \theta} = \frac{f_0'(\xi)}{1 + f_0'(\xi) A},$$
$$\frac{\partial \xi}{\partial A} = \frac{-f_0(\xi)}{1 + f_0'(\xi) A}, \qquad \frac{\partial \xi}{\partial \theta} = \frac{1}{1 + f_0'(\xi) A}. \tag{7}$$

Here the prime means differentiation with respect to ξ. From (7) we obtain

$$\frac{\partial^2 A f(A,\theta)}{\partial A^2} = \frac{1}{(1 + A f_0'(\xi))^3}\big(-2 f_0(\xi) f_0'(\xi) + A\big(-2 f_0(\xi)\big(f_0'(\xi)\big)^2 + f_0(\xi)^2 f_0''(\xi)\big)\big), \tag{8}$$

and

$$\frac{\partial^2 f(A,\theta)}{\partial \theta^2} = \frac{f_0''(\xi)}{(1 + A f_0'(\xi))^3}. \tag{9}$$

By comparison of Eqs. (8) and (9) we see that we can form a linear second order differential equation of the form

$$\frac{\partial^2 A f(A,\theta)}{\partial A^2} = (a + bA)\frac{\partial^2 f(A,\theta)}{\partial \theta^2}, \tag{10}$$

with a and b to be determined, provided

$$a f_0''(\xi) = -2 f_0(\xi) f_0'(\xi), \tag{11}$$

and

$$b f_0''(\xi) = -2 f_0(\xi)\big(f_0'(\xi)\big)^2 + f_0(\xi)^2 f_0''(\xi). \tag{12}$$

To solve, insert Eq. (11) in (12), which gives

$$f_0\big(\xi(A,\theta)\big) = f(A,\theta) = \sqrt{b}\coth\big(\sqrt{b}\,\xi(A,\theta)/a\big). \tag{13}$$

By inserting this result we see that Eqs. (11) and (12) are satisfied without any restrictions on a and b.

It now remains to see whether the result (13) fits with QCD_2, where

$$\rho_{A=0}(\theta) = \sum_{n=-\infty}^{\infty} \delta(\theta - 2\pi n), \tag{14}$$

so from Eq. (3) we get

$$f(A=0,\theta) = f_0(\theta) = \sum_{n=-\infty}^{\infty} \frac{1}{\theta - 2\pi n} = \frac{1}{2}\cot\frac{\theta}{2}. \tag{15}$$

If we compare this with the solution (13) we see that QCD_2 satisfies the linear second order differential equation (10) provided $a = 1$ and $\sqrt{b} = i/2$. Hence the differential equation (10) becomes

$$\frac{\partial^2 A f(A,\theta)}{\partial A^2} - \frac{1}{4}\left(\frac{4}{A} - 1\right)\frac{\partial^2 A f(A,\theta)}{\partial \theta^2} = 0. \tag{16}$$

This linear equation can thus replace the non-linear Hopf equation (5) when the QCD_2 initial value (15) is used.

It should be mentioned that in a similar manner one can produce other second order linear versions of the Hopf equation. For example, instead of Eq. (16) we could study

$$\frac{\partial^2 f(A,\theta)}{\partial A^2} = (c + dA)\frac{\partial^2 f(A,\theta)}{\partial \theta^2}, \tag{17}$$

which leads to a cubic equation for the initial value function

$$c f_0(\xi) - 1/3 f_0(\xi)^3 = d\xi + \text{const}. \tag{18}$$

This equation has no relation with QCD_2, and the linear differential equation (17) must therefore be rejected. Only Eqs. (10) and (16) are relevant.

As shown in Ref. [3] the Hopf equation (or simple generalizations) occurs in any multiplicatively free family of random variables. Thus the method used above to rewrite the Hopf equation in terms of equations which incorporate the initial conditions can be used to obtain a linear replacement of the relevant Hopf equation. The point is then to chose the linear equation such that the initial function is physically relevant, which may not always be possible.

Let us now return to the linear QCD_2 equation (16). Suppose this is all we know, then how do we solve it? The simplest way is to use the factorization $f(A,\theta) = f_1(A) f_2(\theta)$. Eq. (16) can be separated in the two variables, and the resulting equation for $f_2(\theta)$ is trivial and has the solution $f_2(\theta) = \exp(\lambda\theta)$, where λ^2 is the separation constant. Since $f(A,\theta)$ is defined to be periodic in θ with period 2π, it follows that λ should be an integer times i. For each such λ the function f_1 should be proportional to a $W^{(n)}(A)$. Referring to of Eqs. (1) and (3) the separation method gives

$$f(A,\theta) = i\left(\frac{1}{2} + \sum_{n=1}^{\infty} W^{(n)}(A) e^{-in\theta}\right), \tag{19}$$

and the Wilson loops should satisfy the separation equation

$$\frac{d^2 W^{(n)}}{dA^2} + \frac{2}{A}\frac{dW^{(n)}}{dA} + \frac{n^2}{4}\left(\frac{4}{A} - 1\right)W^{(n)} = 0. \tag{20}$$

This equation has the solution

$$W^{(n)}(A) = \frac{1}{n} L_{n-1}^1(nA) e^{-nA/2}, \tag{21}$$

where the L's are the Laguerre polynomials of type 1, and where we used the boundary condition $W^{(n)}(0) = 1$ to fix an

[1] Although the viscosity term usually occurring in Burgers equation is absent here.

arbitrary constant. Eq. (21) is the well-known solution for winding loops [5] in QCD_2.

Eq. (20) can of course be derived directly from the known solution (21). The reason we have started from the Hopf equation (5) is that it can be generalized to higher dimensions as is seen in Eq. (4). In $D = 3$ or 4 it is not true that the resolvent $f_C(\theta)$ only depends on the area, even for simple loops. However, one may speculate that area dependence sets in asymptotically for sufficiently large loops. On the other hand, from the works in Ref. [6] it follows that the loop equation in $D = 4$ has features quite different from its $D = 2$ counterpart. It would be an interesting project for future research to try to invent a generalization of the method of characteristics (leading to Eqs. (6) and (7) in two dimensions) to the functional version of the Hopf equation valid in higher than two dimensions, incorporating the complexities discussed in Ref. [6].

In a recent work by Narayanan and Neuberger [7] they have considered the spectral density in four dimensions and found that it is similar to the density in two dimensions. This approach refers to Wilson loops in which the Coulomb-like contribution has been smeared in order to avoid divergences, thereby making the $D = 4$ case similar to the $D = 2$ case. If the functional Hopf equation (4) could be similarly smeared it would probably be similar to its two-dimensional version for large *smeared* loops.

Eq. (20) is the (zero angular momentum) radial Schrödinger equation for the hydrogen atom if we identify the radial variable with the loop quantity A and the charge with n. Since n quarks are running around on the boundary, this is a quite natural identification. Similarly the "bound state" energy is $-n^2/4$. The point $A = 4$ is essentially the "Bohr radius" of the atom. Inside this radius the "wave function" $W^{(n)}$ oscillates as a function of A, whereas outside this quantity is exponentially damped. Although confinement is often considered trivial in two dimensions, the "wave functions" $W^{(n)}$ behave in a rather non-trivial manner, especially for larger n's. Similar behavior does not occur in the Abelian case in two dimensions. Since there exists the hope that the Hopf equation can be treated by a functional version of the method of characteristics in three or four dimensions, the bound state picture of confinement may be valid also in higher dimensions.

Loop space dynamics thus results in a rather surprising picture of confinement, where the area acts as a radial variable in a three-dimensional space in the sense that the force between the quarks with charges n is governed by the three-dimensional Coulomb potential. The winding Wilson loops $W^{(n)}(A)$ act as wave functions satisfying a Schrödinger equation. Inside the localization radius $A = 4$ these wave functions oscillate (more and more when n increases) and outside $A = 4$ they decay exponentially. These three dimensions are genuinely a product of the loop dynamics, which seems to have forgotten about the two dimensions where it lives. It would be interesting to investigate other more complicated loops to see to what extent this picture of confinement is universal.

One can speculate that the reason for the *localized* bound state picture is that QCD_2 basically is a family of multiplicatively free random variables [3], which is in analogy with Anderson localization of the wave function in the presence of random fields in quantum mechanics. It also could well be that a similar phenomenon occurs in higher dimensions due to the expected random fields in the QCD vacuum.

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